New upper bound for a class of vertex Folkman numbers

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Abstract

Let $a_1 ,\dots, a_r$ be positive integers, $m = \sum_{i=1}^{r}(a_i-1)+1$ and $p = \max\{a_1,\dots,a_r\}$. For a graph G the symbol $G \to \{a_1,\dots,a_r\}$ denotes that in every r-coloring of the vertices of G there exists a monochromatic a_i-clique of color i for some $i = 1,\dots,r$. The vertex Folkman numbers $F(a_1,\dots,a_r; m-1) = \min\{|V(G)| : G \to (a_1 \dots a_r)$ and $K_{m-1} \nsubseteq G\}$ are considered. We prove that $F(a_1,\dots,a_r; m-1) \le m+3p$, $p \ge 3$. This inequality improves the bound for these numbers obtained in [6].

1 Introduction

We consider only finite, non-oriented graphs without loops and multiple edges. We call a p-clique of the graph G a set of p vertices, each two of which are adjacent. The largest positive integer p, such that the graph G contains a p-clique is denoted by $cl(G)$. In this paper we shall also use the following notations:

$V(G)-$ vertex set of the graph G;

$E(G)-$ edge set of the graph G;

$\bar{G}-$ the complement of G;

G[V], $V \subseteq V(G)-$ the subgraph of G induced by V;
$G - V-$ the subgraph induced by the set $V(G)\backslash V$;
$N_G(v), v \in V(G)-$ the set of all vertices of G adjacent to v;
K_n- the complete graph on n vertices;
C_n- simple cycle on n vertices;
P_n- path on n vertices;
$\chi(G)-$ the chromatic number of G;
$\lceil x \rceil-$ the least positive integer greater or equal to x.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] \mid x \in V(G_1), y \in V(G_2)\}$.

Definition *Let $a_1, \ldots, a_r$ be positive integers. We say that the r-coloring*

$$V(G) = V_1 \cup \ldots \cup V_r \ , \ V_i \cap V_j = \emptyset \ , \ i \neq j$$

of the vertices of the graph G is $(a_1, \ldots, a_r)$-free, if V_i does not contain an a_i-clique for each $i \in \{1, \ldots, r\}$. The symbol $G \to (a_1, \ldots, a_r)$ means that there is not $(a_1, \ldots, a_r)$ -free coloring of the vertices of G.

We consider for arbitrary natural numbers $a_1, \ldots, a_r$ and q

$$H(a_1, \ldots a_r; q) = \{G : G \to (a_1, \ldots, a_r) \text{ and } cl(G) < q\}$$

The vertex Folkman numbers are defined by the equalities

$$F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \in H(a_1, \ldots, a_r; q)\}.$$

It is clear that $G \to (a_1, \ldots, a_r)$ implies $cl(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman [3] proved that there exists a graph G such that $G \to (a_1, \ldots, a_r)$ and $cl(G) = \max\{a_1, \ldots, a_r\}$. Therefore

$$F(a_1, \ldots, a_r; q) \text{ exists if and only if } q > \max\{a_1, \ldots, a_r\}. \quad (1)$$

These numbers are called vertex Folkman numbers. In [5] Luczak and Urbansky defined for arbitrary positive integers $a_1, \ldots, a_r$ the numbers

$$m = m(a_1, \ldots, a_r) = \sum_{i=1}^{r}(a_i - 1) + 1 \text{ and } p = p(a_1, \ldots, a_r) = \max\{a_1, \ldots, a_r\}. \quad (2)$$

Obviously $K_m \to (a_1, \ldots, a_r)$ and $K_{m-1} \nrightarrow (a_1, \ldots, a_r)$. Therefore if $q \geq m + 1$ then $F(a_1, \ldots, a_r; q) = m$.

From (1) it follows that the number $F(a_1, \ldots, a_r; q)$ exists if and only if $m \geq p + 1$. Luczak and Urbansky [5] proved that $F(a_1, \ldots, a_r; q) = m + p$. Later ,in [6], Luczak, Rucinsky and Urbansky proved that $K_{m-p-1} + \bar{C}_{2p+1}$ is the only graph in $H(a_1, \ldots, a_r; m)$ with $m + p$ vertices.

From (1) it follows that the number $F(a_1, \dots, a_r; m-1)$ exists if and only if $m \geq p+2$. An overview of the results about the numbers $F(a_1, \dots, a_r; m-1)$ was given in [1]. Here we shall note only the general bounds for the numbers $F(a_1, \dots, a_r; m-1)$. In [8] the following lower bound was proved

$$F(a_1, \dots, a_r; m-1) \geq m+p+2 \ , \ p \geq 2.$$

In the above equality an equality occurs in the case when $\max\{a_1, \dots, a_r\} = 2$ and $m \geq 5$ (see [4,6,7]). For these reasons we shall further consider only the numbers $F(a_1, \dots, a_r; m-1)$ when $\max\{a_1, \dots, a_r\} \geq 3$.

In [6] Luczak, Rucinsky and Urbansky proved the following upper bound for the numbers $F(a_1, \dots, a_r; m-1)$:

$$F(a_1, \dots, a_r; m-1) \leq m+p^2 \ , \text{ for } m \geq 2p+2$$

In [6] they also announced without proof the following inequality :

$$F(a_1, \dots, a_r; m-1) \leq 3p^2+p-mp+2m-3 \ , \text{ for } p+3 \leq m \leq 2p+1.$$

In this paper we shall improve these bounds proving the following

Main theorem *Let $a_1, \dots, a_r$ be positive integers and m and p be defined by (2). Let $m \geq p$ and $p \geq 3$. Then*

$$F(a_1, \dots, a_r; m-1) \leq m+3p.$$

Remark *This bound is exact for the numbers $F(2,2,3;4)$ and $F(3,3;4)$ because*

$$F(2,2,3;4) = 14 \text{ (see [2])}$$

and

$$F(3,3;4) = 14 \text{ (see [9]).}$$

2 Main construction

We consider the cycle C_{2p+1}. We assume that

$$V(C_{2p+1}) = \{v_1, \dots, v_{2p+1}\}$$

and

$$E(C_{2p+1}) = \{[v_i, v_{i+1}], i = 1, \dots, 2p\} \cup \{v_1, v_{2p+1}\}.$$

Let σ denote the cyclic automorphism of C_{2p+1}, i.e. $\sigma(v_i) = v_{i+1}$ for $i = 1, \dots, 2p$, $\sigma(v_{2p+1}) = v_1$.Using this automorphism and the set $M_1 = V(C_{2p+1}) \setminus \{v_1, v_{2p-1}, v_{2p-2}\}$ we

define $M_i = \sigma^{i-1}(M_1)$ for $i = 1, \dots, 2p+1$. Let Γ_p denote the extension of the graph $\bar{C}_{2p+1}$ obtained by adding the new pairwise independent vertices $u_1, \dots, u_{2p+1}$ such that

$$N_{\Gamma_p}(u_i) = M_i \text{ for } i = 1, \dots, 2p+1 \ . \tag{3}$$

We easily see that $cl(\bar{C}_{2p+1}) = p$.

Now we extend σ to an automorphism of Γ_p via the equalities $\sigma(u_i) = u_{i+1}$, for $i = 1, \dots, 2p$, and $\sigma(u_{2p+1}) = u_1$ Now it is clear that

$$\sigma \text{ is an isomorphism of } \Gamma_p. \tag{4}$$

The graph Γ_p was defined for the first time in [8]. In [8] it is also proved that $\Gamma_p \to (3, p)$ for $p \geq 3$. For the proof of the main theorem we shall also use the following generalisation of this fact.

Theorem 1. *Let $p \geq 3$ be a positive integer and $m = p + 2$. Then for arbitrary positive integers $a_1, \dots, a_r$ (r is not fixed) such that*

$$m = 1 + \sum_{i}^{r}(a_i - 1)$$

and $\max\{a_1, \dots, a_r\} \leq p$ *we have*

$$\Gamma_p \to (a_1, \dots a_r).$$

3 Auxiliiary results

The next proposition is well known and easy to prove.

Proposition 1 *Let $a_1, \dots, a_r$ be positive integers and $n = a_1 + \ldots + a_r$. Then*

$$\lceil \frac{a_1}{2} \rceil + \ldots + \lceil \frac{a_r}{2} \rceil \geq \lceil \frac{n}{2} \rceil.$$

If n is even than this inequality is strict unless all the numbers $a_1, \dots, a_r$ are even. If n is odd then this inequality is strict unless exactly one of the numbers $a_1, \dots, a_r$ is odd.

Let P_k be the simple path on k vertices. Let us assume that

$$V(P_k) = \{v_1, \dots, v_k\}$$

and

$$E(P_k) = \{[v_i, v_{i+1}], i = 1, \dots, k-1\}.$$

We shall need the following obvious facts for the complementary graph $\bar{P}_k$ of the graph P_k:

$$cl(\bar{P}_k) = \lceil \frac{k}{2} \rceil \tag{5}$$

$$cl(\bar{P}_{2k} - v) = cl(\bar{P}_{2k}), \text{ for each } v \in V(\bar{P}_{2k}) \tag{6}$$

$$cl(\bar{P}_{2k} - \{v_{2k-2}, v_{2k-1}\}) = cl(\bar{P}_{2k+1}) \text{ for } k \geq 2 \tag{7}$$

$$cl(\bar{P}_{2k+1} - v_{2i}) = cl(\bar{P}_{2k+1}) \ , \ i = 1, \ldots, k \ , \ k \geq 1. \tag{8}$$

The proof of Theorem 1 is based upon three lemmas.

Lemma 1 *Let $V \subset V(C_{2p+1})$ and $|V| = n < 2p+1$. Let $G = \bar{C}_{2p+1}[V]$ and let $G_1, \ldots, G_s$ be the connected components of the graph $\bar{G} = C_{2p+1}[V]$.*

Then

$$cl(G) \geq \lceil \frac{n}{2} \rceil. \tag{9}$$

If n is even, then (9) is strict unless all $|V(G_i)|$ for $i = 1, \ldots, s$ are even. If n is odd, then (9) is strict unless exactly one of the numbers $|V(G_i)|$ is odd.

Proof Let us observe that

$$G = \bar{G}_1 + \ldots + \bar{G}_s. \tag{10}$$

Since $V \neq V(C_{2p+1})$ each of the graphs G_i is a path.From (10) and (5) it follows that

$$cl(G) = \sum_{i=1}^{s} \lceil \frac{n_i}{2} \rceil,$$

where $n_i = |V(G_i)|$, $i = 1, \ldots, s$. From this inequality and Proposition 1 we obtain the inequality (9) . From Proposition 1 it also follows that if n is even then there is equality in (9) if and only if the numbers $n_1, \ldots, n_s$ are even , and if n is odd then we have equality in (9) if and only if exactly one of the numbers $n_1, \ldots, n_s$ is odd.

Corollary 1 *It is true that $cl(\Gamma_p) = p$.*

Proof It is obvious that $cl(\bar{C}_{2p+1}) = p$ and hence $cl(\Gamma_p) \geq p$. Let us denote an arbitrary maximal clique of Γ_p by Q. Let us assume that $|Q| > p$. Then Q must contain a vertex u_i for some $i = 1, \ldots, 2p+1$. As the vertices u_i are pairwise independent Q must contain at most one of them. Since σ is an automorphism of Γ_p (see (4)) and $u_i = \sigma^{i-1}(u_1)$, we may assume that Q contains u_1. Let us assign the subgraph of Γ_p induced by $N_{\Gamma_p(u_1)} = M_1$ by H. The connected components of H are $\{v_2, v_3, \ldots, v_{2p-3}\}$ and $\{v_{2p}, v_{2p+1}\}$ and the both

of them contain an even number of vertices. Using Lemma 1 we have $cl(H) = p - 1$. Hence $|Q| = p$ and this contradicts the assumption .

The next two lemmas follow directly from (10) , (6) , (7) , and (8) and need no proof.

Lemma 2 *Let $V \subsetneq V(C_{2p+1})$ and $G = \bar{C}_{2p+1}[V]$. Let $P_k = \{v_1, v_2, \ldots, v_k\}$ be a connected component of the graph $\bar{G} = C_{2p+1}[V]$. Then*

(a) if $k = 2s$ then

$$cl(G - v_i) = cl(G), i = 1, \ldots, 2s$$

and

$$cl(G - \{v_{2s-2}, v_{2s-1}\}) = cl(G).$$

(b) if $k = 2s + 1$ then

$$cl(G - v_{2i}) = cl(G), i = 1, \ldots, s$$

Lemma 3 *Let $V \subseteq V(C_{2p+1})$ and $\bar{C}_{2p+1} = G$. Let $P_{2k} = \{v_1, \ldots, v_{2k}\}$ and $P_s = \{w_1, \ldots, w_s\}$ be two connected components of the graph $\bar{G} = C_{2p+1}[V]$. Then*

(a) if $s = 2t$ then

$$cl(G - \{v_i, w_j\}) = cl(G),$$

for $i = 1, \ldots, 2k,\ j = 1, \ldots, s$ and

$$cl(G - \{v_{2k-2}, v_{2k-1}, w_j\}) = cl(G),$$

for $j = 1, \ldots, s$.

(b) If $s = 2t + 1$ then

$$cl(G - \{v_{2k-2}, v_{2k-1}, w_{2i}\}) = cl(G)\ ,\ for\ i = 1, \ldots, t.$$

4 Proof of theorem 1

We shall prove Theorem 1 by induction on r. As $m = \sum_{i=1}^{r}(a_i - 1) + 1 = p + 2$ and $\max\{a_1, \ldots, a_r\} \leq p$ we have $r \geq 2$. Therefore the base of the induction is $r = 2$. We warn the reader that the proof of the inductive base is much more involved then the proof of the inductive step. Let $r = 2$ and $(a_1 - 1) + (a_2 - 1) + 1 = p + 2$ and $\max\{a_1, a_2\} \leq p$. Then we have

$$a_1 + a_2 = p + 3. \tag{11}$$

Since $p \geq 3$ and $\max\{a_1, a_2\} \leq p$ we have that

$$a_i \geq 3\ ,\ i = 1, 2. \tag{12}$$

We must prove that $\Gamma_p \to (a_1, a_2)$. Assume the opposite and let $V(\Gamma_p) = V_1 \cup V_2$ be a (a_1, a_2)-free coloring of $V(\Gamma_p)$. Define the sets

$$V_i' = V_i \cap V(\bar{C}_{2p+1})\ ,\ i = 1, 2$$

and the graphs

$$G_i = \bar{C}_{2p+1}[V_i'] \ , \ i = 1, 2.$$

By assumption V_i does not contain an a_i-clique and hence V_i' does not contain an a_i-clique, too. Therefore from Lemma 1 we have $V_i' \leq 2a_i - 2$, i=1,2. From these inequalities and the equality

$$|V_1'| + |V_2'| = 2p + 1 = 2a_1 + 2a_2 - 5$$

(as $p = a_1 + a_2 - 3-$ see (11)) we have two possibilities:

$$|V_1'| = 2a_1 - 2 \ , \ |V_2'| = 2a_2 - 3$$

or

$$|V_1'| = 2a_1 - 3 \ ,|V_2'| = 2a_2 - 2.$$

Without loss of generality we assume that

$$|V_1'| = 2a_1 - 2 \ , \ |V_2'| = 2a_2 - 3. \tag{13}$$

From (13) and Lemma 1 we obtain $cl(G_i) \geq a_i - 1$ and by the assumption that the coloring $V_1 \cup V_2$ is (a_1, a_2)-free we have

$$cl(G_i) = a_i - 1 \text{ for } i = 1, 2. \tag{14}$$

From (13) , (14) and Lemma 1 we conclude that

The number of the vertices of each connected component of $\bar{G}_1$ is an even number; (15)

and

the number of the vertices of exactly one of the connected components of $\bar{G}_2$ is an odd number. (16)

According to (15) there are two possible cases.

Case 1. Some connected component of $\bar{G}_1$ has more then two vertices. Now from (15) it follows that this component has at least four vertices. Taking into consideration (15) and (4) we may assume that $\{v_1, \ldots, v_{2s}\}$, $s \geq 2$ is a connected component of $\bar{G}_1$. Since V_1' does not contain an a_1-clique we have by Lemma 1 that $s < a_1$. Therefore $2s + 2 \leq 2p$ and we can consider the vertex u_{2s+2}.

Subcase 1.a. Assume that $u_{2s+2} \in V_1$. Let $v_{2s+2} \in V_2'$. We have from (3) that

$$N_{\Gamma_p}(u_{2s+2}) \supseteq V_1' - \{v_{2s-2}, v_{2s-1}\}. \tag{17}$$

From (14) and Lemma 2 (a) we have that $V_1' - \{v_{2s-2}, v_{2s-1}\}$ contains an $(a_1 - 1)$-clique Q . From (17) it follows that $Q \cup \{u_{2s+2}\}$ is an a_1-clique in V_1 which is a contradiction.

Now let $v_{2s+2} \in V'_1$. From (3) we have

$$N_{\Gamma_P}(u_{2s+2}) \supseteq V'_1 - \{v_{2s-2}, v_{2s-1}, v_{2s+2}\}. \tag{18}$$

According to (15) we can apply Lemma 3 (a) for the connected component $\{v_1, \ldots, v_{2s}\}$ of $\bar{G}_1$ and the connected component of $\bar{G}_1$ that contains v_{2s+2}. We see from (14) and Lemma 3(a) that $V'_1 - \{v_{2s-2}, v_{2s-1}, v_{2s+2}\}$ contains an $(a_1 - 1)$-clique Q of the graph G_1. Now from (18) it follows that $Q \cup \{u_{2s+2}\}$ is an a_1-clique in V_1, which is a contradiction.

Subcase 1.b. Assume that $u_{2s+2} \in V_2$. If $v_{2s+2} \notin V'_2$ then from (3) it follows

$$N_{\Gamma_p}(u_{2s+2}) \supseteq V'_2. \tag{19}$$

As V'_2 contains an $(a_2 - 1)$-clique Q (see (14)). From (19) it follows that $Q \cup \{u_{2s+2}\}$ is an a_2-clique in V_2, which is a contradiction.

Let now $v_{2s+2} \in V'_2$. In this situation we have from (3)

$$N_{\Gamma_p}(u_{2s+2}) \supseteq V'_2 - \{v_{2s+2}\}. \tag{20}$$

We shall prove that

$$V_2 - \{v_{2s+2}\} \text{ contains an } (a_2 - 1)\text{-clique of } \Gamma_p. \tag{21}$$

As v_{2s} is the last vertex in the connected component of G_1, we have $v_{2s+1} \in V'_2$. Let L be the connected component of $\bar{G}_2$ containing v_{2s+2}. Now we have $L = \{v_{2s+1}, v_{2s+2}, \ldots\}$. Now (21) follows from Lemma 2(b) applied to the component L. From (20) and(21) it follows that V_2 contains aa a_2-clique, which is a contradiction.

Case 2. Let some connected component of $\bar{G}_1$ have exactly two vertices.

From (12) and (13) it follows that $\bar{G}_1$ has at least two connected components. It is clear that $\bar{G}_2$ also has at least two components. From (16) we have that the number of the vertices of at least one of the components of G_2 is even. From these considerations and (4) it follows that it is enough to consider the situation when $\{v_1, v_2\}$ is a connected component of $\bar{G}_1$ and $\{v_3, \ldots, v_{2s}\}$ is a component of $\bar{G}_2$, and $\{v_{2s+1}, v_{2s+2}\}$ is a component of $\bar{G}_1$. We shall consider two subcases.

Subcase 2.a. If $u_{2s+2} \in V_1$.

Let $s = 2$. We apply Lemma 3(a) to the components $\{v_1, v_2\}$ and $\{v_5, v_6\}$. From (14) we conclude that

$$V'_1 - \{v_2, v_6\} \text{ contains an } (a_1 - 1)\text{-clique}. \tag{22}$$

From (3) we have

$$N_{\Gamma_p}(u_6) \supseteq V'_1 - \{v_2, v_6\}. \tag{23}$$

Now (22) and (23) give that V_1 contains an a_1-clique.

Let $s \geq 3$. From (3) we have

$$N_{\Gamma_p}(u_{2s+2}) \supseteq V_1' - \{v_{2s+2}\}. \tag{24}$$

According to Lemma 2(a) $V_1' - \{v_{2s+2}\}$ contains an $(a_1 - 1)$-clique. Now using (24) we have that this $(a_1 - 1)$-clique together with the vertex u_{2s+2} gives an a_1-clique in V_1. Subcase 2.a. is proved.

Subcase 2.b. Let $u_{2s+2} \in V_2$.

Let $s = 2$. From (3) we have $N_{\Gamma_p}(u_6) \supseteq V_2' - \{v_3\}$. According to Lemma 2(a) and (14) $V_2' - \{v_3\}$ contains an $(a_2 - 1)$-clique. This clique together with $u_{2s+2} \in V_2$ gives an a_2-clique in V_2, which is a contradiction.

Let $s \geq 3$. Here from (3) we have $N_{\Gamma_p}(u_{2s+2}) \supseteq V_2' - \{v_{2s-2}, v_{2s-1}\}$. According to Lemma 2 (a) and (14) we have that $V_2' - \{v_{2s-2}, v_{2s-1}\}$ contains an $(a_2 - 1)$-clique. This clique together with $u_{2s+2} \in V_2$ gives an a_2-clique in V_2, which is a contradiction. This completes the proof of case 2 and of the inductive base $r = 2$.

Now we more easily handle the case $r \geq 3$. It is clear that

$$G \to (a_1, \ldots, a_r) \Leftrightarrow G \to (a_{\varphi(1)}, \ldots, a_{\varphi(r)})$$

for any permutation $\varphi \in S_r$. That is why we may assume that

$$a_1 \leq \ldots \leq a_r \leq p. \tag{25}$$

We shall prove that $a_1 + a_2 - 1 \leq p$. If $a_2 \leq 2$ this is trivial : $a_1 + a_2 - 1 \leq 3 \leq p$. Let $a_2 \geq 3$. From (25) we have $a_i \geq 3$, $i = 2, \ldots, r$. From these inequalities and the statement of the theorem

$$\sum_{i=1}^{r}(a_i - 1) + 1 = p + 2$$

we have

$$p + 2 \geq 1 + (a_2 - 1) + (a_1 - 1) + 2(r - 2).$$

From this inequality and $r \geq 3$ it follows that $a_1 + a_2 - 1 \leq p$. Thus we can now use the inductive assumption and obtain

$$\Gamma_p \to (a_1 + a_2 - 1, a_3, \ldots, a_r). \tag{26}$$

Consider an arbitrary r-coloring $V_1 \cup \ldots \cup V_r$ of $V(\Gamma_p)$. Let us assume that V_i does not contain an a_i-clique for each $i = 3, \ldots, r$. Then from (26) we have $V_1 \cup V_2$ contains $(a_1 + a_2 - 1)$-clique. Now from the pigeonhole principle it follows that either V_1 contains an a_1-clique or V_2 contains an a_2-clique. This completes the proof of Theorem 1.

5 Proof of the main theorem

Let m and p be positive integers $p \geq 3$ and $m \geq p + 2$. We shall first prove that for arbitrary positive integers $a_1, \ldots, a_r$ such that

$$m = 1 + \sum_{i=1}^{r}(a_i - 1)$$

and $\max\{a_1, \dots, a_r\} \leq p$ we have

$$K_{m-p-2} + \Gamma_p \to (a_1, \dots, a_r). \tag{27}$$

We shall prove (27) by induction on $t = m - p - 2$. As $m \geq p + 2$ the base is $t = 0$ and it follows from Theorem 1. Assume now $t \geq 1$. Then obviously

$$K_{m-p-2} + \Gamma_p = K_1 + (K_{m-p-3} + \Gamma_p).$$

Let $V(K_1) = \{w\}$. Consider an arbitrary r-coloring $V_1 \cup \dots \cup V_r$ of $V(K_{m-p-2} + \Gamma_p)$. Let $w \in V_i$ and V_j , $j \neq i$ does not contain a a_j-clique.

In order to prove (27) we need to prove that V_i contains an a_i-clique. If $a_i = 1$ this is clear as $w \in V_i$. Let $a_i \geq 2$. According to the inductive hypothesis we have

$$K_{m-p-3} + \Gamma_p \to (a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_r). \tag{28}$$

We consider the coloring

$$V_1 \cup \dots \cup V_{i-1} \cup \{V_i - w\} \cup \dots \cup V_r$$

of $V(K_{m-p-3}+\Gamma_p)$. As V_j, $j \neq i$ do not contain a_j-cliques, from (28) we have that $V_i - \{w\}$ contains an $(a_i - 1)$-clique. This $(a_i - 1)$-clique together with w form an a_i-clique in V_i. Thus (27) is proved.

From Corollary 1 obviously follows that $cl(K_{m-p-2} + \Gamma_p) = m - 2$. From this and (27) we have $K_{m-p-2} + \Gamma_p \in H(a_1, \dots, a_r; m - 1)$. The number of the vertices of the graph $K_{m-p-2} + \Gamma_p$ is $m + 3p$ therefore $F(a_1, \dots, a_r; m - 1) \leq m + 3p$.

The main theorem is proved.

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